The Geometry of Autonomous Metrical Multi-Time Lagrange Space of Electrodynamics

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Abstract

Section 1 contains physical and geometrical aspects that motivates us to study the autonomous multi-time Lagrangian space of electrodynamics. Section 2 constructs the canonical nonlinear connection Γ and the Cartan canonical Γ-linear connection of this space. Section 3 describes the Maxwell equations which govern the electromagnetic field of this space. The Einstein equations of gravitational potentials of the autonomous multi-time Lagrange space are written in Section 4. The conservation laws of these equations will be also described.

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1 Geometrical and physical aspects

In the last thirty years, many geometrical models in Mechanics or Physics were based on the notion of ordinary Lagrangian. Thus, the geometrical concept of Lagrange space was introduced. The differential geometry of the Lagrange spaces is now considerably developped and used in various fields to study the natural processes where the dependence on position, velocity or momentum are involved [5]. We recall that a *Lagrange space* $L^n = (M, L(x, y))$ is defined as a pair which consists of a real, n-dimensional manifold M coordinated by $x = (x^i)_{i=\overline{1,n}}$ and a *regular* Lagrangian $L : TM \to R$ (i. e. the fundamental metrical d-tensor $g_{ij}(x, y) = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial y^i \partial y^j}$ is of rank n and has a constante signature on $TM\backslash\{0\}$). We point out that the Lagrangian L is not necessarily homogenous with respect to the direction $y = (y^i)_{i=\overline{1,n}}$.

An important and well known example of Lagrange space comes from electrodynamics. Thus, let us consider the Lagrangian $L : TM \to R$ which governs the movement law of a particle of mass $m \neq 0$ and electric charge e, placed concomitantly into a gravitational field and an electromagnetic one,

$$L(x, y) = mc\varphi_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + U(x), \tag{1.1}$$

where the pseudo-Riemannian metric $\varphi_{ij}(x)$ represents the *gravitational potentials* of the space M, $A_i(x)$ are the components of a covector field on M representing the *electromagnetic potentials*, $U(x)$ is a function on M which is called *potential*

function and c is the physical constant of light speed. It is obvious that L is a regular Lagrangian and, consequently, the pair $L^n = (M, L)$ is a Lagrange space. This space is known under the name of *the Lagrange space of electrodynamics.*

At the same time, there are many problems in Physics and Variational Calculus in which multi-time Lagrangians functions L depending of first order partial derivatives or, alternatively, of partial directions, are involved. In this context, the Lagrangian function L is defined on the total space of the 1-jet fibre bundle $J^1(T, M)$, where T is a smooth, real, p-dimensional manifold coordinated by $t = (t^\alpha)_{\alpha=\overline{1,p}}$, whose physical meaning is that of *"multidimensional time"*. We point out that $J^1(T, M)$ is coordinated by $(t^\alpha, x^i, x^i_\alpha)$.

It is well known that the jet fibre bundle of order one is a basic object in the study of classical and quantum field theories. From a certain physical point of view, the 1-jet fibre bundle $J^1(T, M) \to T \times M$ is regarded as a *bundle of configurations* [3], [10], [11].

Let us consider the particular case $T = R$ (i. e. the usual time axis represented by the set of real numbers) in the construction of jet bundle of order one, we find the bundle

$$J^1(R, M) \equiv R \times TM \to R \times M, \quad (t, x^i, y^i) \to (t, x^i), \tag{1.2}$$

that is, the bundle of configurations of *relativistic rheonomic mechanics* whose invariance gauge group is [10]

$$\begin{cases} \tilde{t} = \tilde{t}(t) \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j}\dfrac{dt}{d\tilde{t}}y^j, \end{cases} \tag{1.3}$$

where the coordinates (t^1, x^i, x^i_1) of the jet space $J^1(R, M) \equiv R \times TM$ are denoted by (t, x^i, y^i). It is obvious that the form of this gauge group emphasizes the *relativistic* character of the time coordinate t.

We underline that, in the *classical rheonomic mechanics* studied in [5], the bundle of configurations is

$$R \times TM \to M, \quad (t, x^i, y^i) \to (x^i), \tag{1.4}$$

whose geometrical invariance group is of the form

$$\begin{cases} \tilde{t} = t \\ \tilde{x}^i = \tilde{x}^i(x^j) \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j}y^j, \end{cases} \tag{1.5}$$

that is, it ignores the temporal reparametrizations. Consequently, in that case, the temporal coordinate t has a character of *absolute* time.

We emphasize that, in the relativistic rheonomic mechanics, a basic role is played by the following Lagrangian of *relativistic rheonomic electrodynamics*,

$$\mathcal{L} = \left[mc\psi^{11}(t)\varphi_{ij}(x)y^i y^j + \frac{2e}{m}A^{(1)}_{(i)}(t, x)y^i + U(t, x) \right] \sqrt{|\psi_{11}|}, \tag{1.6}$$

where ψ_{11} is a semi-Riemannian metric on R, $A^{(1)}_{(i)}(t, x)$ is a distiguished tensor on $J^1(R, M)$ and $U(t, x)$ is a smooth function on $R \times M$.

At the same time, we point out that the Lagrangian which governs the *classical rheonomic electrodynamics* has the form

$$L = mc\varphi_{ij}(x)y^i y^j + \frac{2e}{m} A_i(t,x)y^i + U(t,x). \tag{1.7}$$

It is important to note the difference between the notions of Lagrangian used in both relativistic and classical rheonomic mechanics. From this point of view, the reader is invited to compare them, following the expositions done in [5], [10]. Thus, according to Olver terminology [4], we point out that, in the background of relativistic rheonomic mechanics, a Lagrangian $\mathcal{L}$ on $J^1(R,M)$ is a local function on the 1-jet space, which transforms by the rule $\tilde{\mathcal{L}} = \mathcal{L}|dt/d\tilde{t}|$. The notion of Lagrangian function L (i. e. , a smooth function $L : J^1(R,M) \to R$) is also involved in relativistic rheonomic mechanics. We point out that, in that case, the geometrical invariance group of the bundle of configurations $J^1(R,M) \to R\times M$ is 1.3. In contrast, in the classical rheonomic mechanics, a Lagrangian L is only a smooth function on the total space of the bundle $R\times TM \to M$. We remark that, in that case, the bundle of configurations has the geometrical invariance group 1.5.

Now, returning us to the general multi-temporal context, we point out that a fundamental geometrical concept used in the geometrization of a multi-time Lagrangian is that of metrical multi-time Lagrange space, introduced in [12]. The differential geometry of metrical multi-time Lagrange spaces is now considerably developped in [8], [12].

In order to develope this geometrical approach, we fix a semi-Riemannian metric $\psi = \psi_{\alpha\beta}(t^\gamma)$ on the temporal manifold T. We recall that a *metrical multi-time Lagrange space* is a pair $ML^n_p = (J^1(T,M), L)$ consisting of 1-jet space and a *Kronecker ψ-regular* multi-time Lagrange function L, that is [12]

$$G^{(\alpha)(\beta)}_{(i)(j)}(t^\gamma, x^k, x^k_\gamma) = \frac{1}{2}\frac{\partial^2 L}{\partial x^i_\alpha \partial x^j_\beta} = \psi^{\alpha\beta}(t^\gamma)\varphi_{ij}(t^\gamma, x^k, x^k_\gamma), \tag{1.8}$$

where $\varphi_{ij}(t^\gamma, x^k, x^k_\gamma)$ is a d-tensor on $J^1(T,M)$, symmetric, of rank n and having a constant signature.

An important example of metrical multi-time Lagrange space, which comes from physics, is offered by the *"energy"* Lagrangian function L used in the Polyakov model of bosonic strings,

$$L(t^\gamma, x^k, x^k_\gamma) = \frac{1}{2}\psi^{\alpha\beta}(t)\varphi_{ij}(x)x^i_\alpha x^j_\beta. \tag{1.9}$$

We recall that the extremals of the Lagrangian $\mathcal{L} = L\sqrt{|\psi|}$ are exactly the harmonic maps between the semi-Riemannian spaces (T,ψ) and (M,φ).

By a natural extension of previous examples of Lagrangian functions, we can offer another important example of metrical multi-time Lagrange space, considering the general Lagrangian function L which comes from electrodynamics and theory of bosonic strings, namely,

$$L = mc\psi^{\alpha\beta}(t)\varphi_{ij}(x)x^i_\alpha x^j_\beta + \frac{2e}{m}A^{(\alpha)}_{(i)}(t,x)x^i_\alpha + U(t,x), \tag{1.10}$$

where $A^{(\alpha)}_{(i)}(t,x)$ is a distiguished tensor on $J^1(T,M)$ and $U(t,x)$ is a smooth function on $T \times M$.

In this context, in order to unify all Lagrangian entities exposed above, we introduce the following

Definition 1.1 The pair $EDML^n_p = (J^1(T,M), L)$ which consists of jet fibre bundle of order one and a Lagrangian function of the form

$$L(t^\gamma, x^k, x^k_\gamma) = h^{\alpha\beta}(t^\gamma) g_{ij}(x^k) x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t^\gamma, x^k) x^i_\alpha + F(t^\gamma, x^k) \tag{1.11}$$

where $h_{\alpha\beta}(t^\gamma)$ (resp. $g_{ij}(x^k)$) is a semi-Riemannian metric on the temporal (resp. spatial) manifold T (resp. M), $U^{(i)}_{(\alpha)}(t^\gamma, x^k)$ are the local components of a distinguished tensor on $J^1(T,M)$ and $F(t^\gamma, x^k)$ is a smooth function on the product manifold $T \times M$, is called an *autonomous metrical multi-time Lagrange space of electrodynamics.*

Remark 1.1 We point out that the non-dynamical character (i. e. , the independence with respect to the temporal coordinates) of the spatial metric $g_{ij}(x^k)$ determined us to use the terminology of *autonomous* in the previous definition.

2 The geometry of autonomous metrical multi-time Lagrange space of electrodynamics

In this section, we will apply the general geometrical development of a metrical multi-time Lagrange space [12], to the particular space of electrodynamics $EDML^n_p$.

In order to do this development, let us consider the energy action functional associated to the multi-time Lagrangian of electrodynamics

$$\mathcal{L} = L\sqrt{|h|} = \left[h^{\alpha\beta}(t^\gamma) g_{ij}(x^k) x^i_\alpha x^j_\beta + U^{(\alpha)}_{(i)}(t^\gamma, x^k) x^i_\alpha + F(t^\gamma, x^k)\right]\sqrt{|h|}, \tag{2.1}$$

namely,

$$\mathcal{E}_\mathcal{L} : C^\infty(T,M) \to R, \quad \mathcal{E}_\mathcal{L}(f) = \int_T \mathcal{L} dt^1 \wedge dt^2 \ldots \wedge dt^p, \tag{2.2}$$

where the temporal manifold T is considered compact and orientable, the local expression of the smooth map f is $(t^\alpha) \to (x^i(t^\alpha))$ and $x^i_\alpha = \partial x^i / \partial t^\alpha$. In this context, it is proved in [12]

Theorem 2.1 *The extremals of the energy functional $\mathcal{E}_L$ associated to the multi-time Lagrangian $\mathcal{L}$ are harmonic maps [11] of the spray (H,G) defined by the temporal components*

$$H^{(i)}_{(\alpha)\beta} = -\frac{1}{2} H^\gamma_{\alpha\beta} x^i_\gamma$$

and the local spatial components

$$G^{(i)}_{(\alpha)\beta} = \frac{1}{2}\gamma^i_{jk} x^j_\alpha x^k_\beta + \frac{h_{\alpha\beta} g^{il}}{4p}\left[U^{(\mu)}_{(l)m} x^m_\mu + \frac{\partial U^{(\mu)}_{(l)}}{\partial t^\mu} + U^{(\mu)}_{(l)} H^\gamma_{\mu\gamma} - \frac{\partial F}{\partial x^l}\right],$$

where $H^{\gamma}_{\alpha\beta}$ (resp. γ^i_{jk}) are the Christoffel symbols of the metric $h_{\alpha\beta}$ (resp. g_{ij}), $p = \dim T$, and $U^{(\alpha)}_{(i)j} = \dfrac{\partial U^{(\alpha)}_{(i)}}{\partial x^j} - \dfrac{\partial U^{(\alpha)}_{(j)}}{\partial x^i}$. In other words, these extremals verify the harmonic map equations attached to the spray (H, G), namely,

$$h^{\alpha\beta}\left\{x^i_{\alpha\beta} + 2H^{(i)}_{(\alpha)\beta} + 2G^{(i)}_{(\alpha)\beta}\right\} = 0. \tag{2.3}$$

Definition 2.1 The spray (H, G) constructed in the previous theorem is called the *canonical spray attached to the autonomous metrical multi-time Lagrange space of electrodynamics.*

Using the canonical spray (H, G), one naturally induces a nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ on $J^1(T, M)$, which is also called the *canonical nonlinear connection of the autonomous metrical multi-time Lagrange space of electrodynamics.* Thus, denoting $\mathcal{G}^i = h^{\alpha\beta}G^{(i)}_{(\alpha)\beta}$, we establish the folowing theorem [12]

Theorem 2.2 *The canonical nonlinear connection of the autonomous metrical multi-time Lagrange space of electrodynamics is determined by the temporal components*

$$M^{(i)}_{(\alpha)\beta} = 2H^{(i)}_{(\alpha)\beta} = -H^{\gamma}_{\alpha\beta}x^i_{\gamma} \tag{2.4}$$

and the local spatial components

$$N^{(i)}_{(\alpha)j} = \frac{\partial \mathcal{G}^i}{\partial x^j_{\gamma}}h_{\alpha\gamma} = \gamma^i_{jk}x^k_{\alpha} + \frac{h_{\alpha\gamma}g^{il}}{4}U^{(\gamma)}_{(l)j}. \tag{2.5}$$

Following the paper [12], by a direct calculation, we determine the *Cartan canonical connection* of the autonomous metrical multi-time Lagrange space of electrodynamics, together with its torsion and curvature local d-tensors.

Theorem 2.3 *i) The Cartan canonical connection $C\Gamma = (H^{\gamma}_{\alpha\beta}, G^k_{j\gamma}, L^i_{jk}, C^{i(\gamma)}_{j(k)})$ of the autonomous metrical multi-time Lagrange space of electrodynamics has the coefficients*

$$H^{\gamma}_{\alpha\beta} = H^{\gamma}_{\alpha\beta}, \quad G^k_{j\gamma} = 0, \quad L^i_{jk} = \gamma^i_{jk}, \quad C^{i(\gamma)}_{j(k)} = 0. \tag{2.6}$$

ii) The torsion $\mathbf{T}$ of the Cartan canonical connection of the autonomous metrical multi-time Lagrange space of electrodynamics is determined by three local d-tensors, namely,

$$\begin{gathered} R^{(m)}_{(\mu)\alpha\beta} = -H^{\gamma}_{\mu\alpha\beta}x^m_{\gamma}, \quad R^{(m)}_{(\mu)\alpha j} = -\frac{h_{\mu\eta}g^{mk}}{4}\left[H^{\eta}_{\alpha\gamma}U^{(\gamma)}_{(k)j} + \frac{\partial U^{(\eta)}_{(k)j}}{\partial t^{\alpha}}\right], \\ R^{(m)}_{(\mu)ij} = r^m_{ijk}x^k_{\mu} + \frac{h_{\mu\eta}g^{mk}}{4}\left[U^{(\eta)}_{(k)i|j} + U^{(\eta)}_{(k)j|i}\right], \end{gathered} \tag{2.7}$$

where $H^{\gamma}_{\mu\alpha\beta}$ (resp. r^m_{ijk}) are the local curvature tensors of the semi-Riemannian metric $h_{\alpha\beta}$ (resp. g_{ij}) and "$_{|i}$" represents the local spatial horizontal covariant derivative induced by the Cartan connection.

iii) The curvature $\mathbf{R}$ of the Cartan canonical connection of the autonomous metrical multi-time Lagrange space of electrodynamics is determined by two local d-tensors, namely, $H^{\eta}_{\alpha\beta\gamma}$ and $R^l_{ijk} = r^l_{ijk}$, that is, exactly the curvature tensors of the semi-Riemannian metrics $h_{\alpha\beta}$ and g_{ij}.

3 Maxwell equations of autonomous metrical multi-time Lagrange space of electrodynamics

In order to develope the electromagnetic theory of the autonomous metrical multi-time Lagrange space, let us consider the *canonical Liouville d-tensor* $\mathbf{C}=x^i_\alpha \dfrac{\partial}{\partial x^i_\alpha}$ on the jet fibre bundle of order one, and let us construct the *deflection d-tensors* [8]

$$
\text{(3.1)}\qquad \begin{cases} \bar{D}^{(i)}_{(\alpha)\beta} = x^i_{\alpha/\beta} = 0 \\ D^{(i)}_{(\alpha)j} = x^i_{\alpha|j} = -\dfrac{1}{4} g^{im} h_{\alpha\mu} U^{(\mu)}_{(m)j} \\ d^{(i)(\beta)}_{(\alpha)(j)} = x^i_\alpha|^{(\beta)}_{(j)} = \delta^i_j \delta^\beta_\alpha, \end{cases}
$$

where "$_{/\beta}$", "$_{|j}$" and "$|^{(\beta)}_{(j)}$" are the local covariant derivatives induced by the Cartan canonical connection $C\Gamma$.

Multiplying the deflection d-tensors by the vertical fundamental metrical d-tensor $h^{\alpha\beta}(t)g_{ik}(x)$, we construct the d-tensors,

$$
\text{(3.2)}\qquad \begin{cases} \bar{D}^{(\alpha)}_{(i)\beta} = 0 \\ D^{(\alpha)}_{(i)j} = -\dfrac{1}{4} U^{(\alpha)}_{(i)j} \\ d^{(\alpha)(\beta)}_{(i)(j)} = h^{\alpha\beta} g_{ij}. \end{cases}
$$

Taking into account the general expressions of the local *electromagnetic d-tensors* of a metrical multi-time Lagrange space [8], by a direct calculation, we deduce the following

Proposition 3.1 *The local electromagnetic d-tensors of the autonomous metrical multi-time Lagrange space of electrodynamics have the expressions,*

$$
\text{(3.3)}\qquad \begin{cases} F^{(\alpha)}_{(i)j} = \dfrac{1}{2}\left[D^{(\alpha)}_{(i)j} - D^{(\alpha)}_{(j)i}\right] = \dfrac{1}{8}\left[U^{(\alpha)}_{(j)i} - U^{(\alpha)}_{(i)j}\right] = -\dfrac{1}{4} U^{(\alpha)}_{(i)j} \\ f^{(\alpha)(\beta)}_{(i)(j)} = \dfrac{1}{2}\left[d^{(\alpha)(\beta)}_{(i)(j)} - d^{(\alpha)(\beta)}_{(j)(i)}\right] = 0. \end{cases}
$$

Particularizing the Maxwell equations of electromagnetic field, described in the general case of a metrical multi-time Lagrange space [8], we deduce the main result of the electromagnetism of the autonomous metrical multi-time Lagrange space of electrodynamics.

Theorem 3.2 *The electromagnetic local components $F^{(\alpha)}_{(i)j}$ of the autonomous metrical multi-time Lagrange space of electrodynamics are governed by the following equations of Maxwell type,*

$$
\text{(3.4)}\qquad \begin{cases} F^{(\alpha)}_{(i)j/\beta} = \dfrac{1}{2} \mathcal{A}_{\{i,j\}} h^{\alpha\mu} g_{im} R^{(m)}_{(\mu)\beta j} \\ \sum_{\{i,j,k\}} F^{(\alpha)}_{(i)j|k} = 0 \\ \sum_{\{i,j,k\}} F^{(\alpha)}_{(i)j}|^{(\gamma)}_{(k)} = 0, \end{cases}
$$

where $\mathcal{A}_{\{i,j\}}$ represents an alternate sum and $\sum_{\{i,j,k\}}$ means a cyclic sum.

4 Einstein equations and conservation laws of autonomous metrical multi-time Lagrange space of electrodynamics

In order to develope the gravitational theory of the autonomous metrical multi-time Lagrange space of electrodynamics $EDML_p^n$, we point out that the vertical metrical d-tensor $G^{(\alpha)(\beta)}_{(i)(j)} = h^{\alpha\beta}(t)g_{ij}(x)$ and the canonical nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$ of this space induce a natural *gravitational h-potential* on the 1-jet space $J^1(T,M)$, which is expressed by [8]

$$G = h_{\alpha\beta}dt^\alpha \otimes dt^\beta + g_{ij}dx^i \otimes dx^j + h^{\alpha\beta}g_{ij}\delta x^i_\alpha \otimes \delta x^j_\beta.$$

Let us consider $C\Gamma = (H^\gamma_{\alpha\beta}, G^k_{j\gamma}, L^i_{jk}, C^{i(\gamma)}_{j(k)})$ the Cartan canonical connection of ML^n_p.

We postulate that the Einstein which govern the gravitational h-potential G of the metrical multi-time Lagrange space of electrodynamics $EDML_p^n$ are the Einstein equations attached to the Cartan canonical connection and the adapted metric G on $J^1(T,M)$, that is,

$$Ric(C) - \frac{Sc(C)}{2}G = \mathcal{K}\mathcal{T},$$

where $Ric(C)$ represents the Ricci d-tensor of the Cartan connection, $Sc(C)$ is its scalar curvature, $\mathcal{K}$ is the Einstein constant and $\mathcal{T}$ is an intrinsec tensor of matter which is called the *stress-energy* d-tensor.

In an adapted basis $(X_A) = \left(\frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^i_\alpha}\right)$, the curvature d-tensor $\mathbf{R}$ of the Cartan connection is expressed locally by $\mathbf{R}(X_C, X_B)X_A = R^D_{ABC}X_d$. Hence, it follows that we have $R_{AB} = Ric(X_A, X_B) = R^D_{ABD}$ and $Sc(C) = G^{AB}R_{AB}$, where

$$G^{AB} = \begin{cases} h_{\alpha\beta}, & \text{for } A = \alpha,\ B = \beta \\ g^{ij}, & \text{for } A = i,\ B = j \\ h_{\alpha\beta}g^{ij}, & \text{for } A = {}^{(i)}_{(\alpha)},\ B = {}^{(j)}_{(\beta)} \\ 0, & \text{otherwise.} \end{cases}$$

Taking into account the expressions of the local curvature d-tensors of the Cartan connection, we deduce that we have the following two effective local Ricci d-tensors, namely, $H_{\alpha\beta}$ and $R_{ij} = r_{ij}$, where $H_{\alpha\beta}$ (resp. r_{ij}) are the Ricci tensors associated to the semi-Riemannian metric $h_{\alpha\beta}$ (resp. g_{ij}), the rest of these vanishing.

Denoting $H = h^{\alpha\beta}H_{\alpha\beta}$, $R = g^{ij}R_{ij}$ and $S = h_{\alpha\beta}g^{ij}R^{(\alpha)(\beta)}_{(i)(j)}$, the scalar curvature of Cartan connection becomes $Sc(C) = H + R + S$. By a direct calculation, we conclude that the effective scalar curvatures of a metrical multi-time Lagrange space are $H = h^{\alpha\beta}H_{\alpha\beta}$ and $r = g^{ij}r_{ij}$, that is, exactly the scalar curvatures of the semi-Riemannian metrics $h_{\alpha\beta}$ and g_{ij}.

Consequently, we can establish the following

Theorem 4.1 *The Einstein equations which govern the gravitational h-potential G, induced by the Lagrangian function of autonomous metrical multi-time Lagrange space of electrodynamics, take the form*

$$(E_1) \qquad \begin{cases} H_{\alpha\beta} - \dfrac{H+r}{2}h_{\alpha\beta} = \mathcal{K}\mathcal{T}_{\alpha\beta} \\ r_{ij} - \dfrac{H+r}{2}g_{ij} = \mathcal{K}\mathcal{T}_{ij} \\ -\dfrac{H+r}{2}h^{\alpha\beta}g_{ij} = \mathcal{K}\mathcal{T}^{(\alpha)(\beta)}_{(i)(j)}, \end{cases}$$

$$(E_2) \qquad \begin{cases} 0 = \mathcal{T}_{\alpha i}, \quad 0 = \mathcal{T}_{i\alpha}, \quad 0 = \mathcal{T}^{(\alpha)}_{(i)\beta} \\ 0 = \mathcal{T}_{\alpha(i)}^{\ \ (\beta)}, \quad 0 = \mathcal{T}_{i(j)}^{\ \ (\alpha)}, \quad 0 = \mathcal{T}^{(\alpha)}_{(i)j}. \end{cases}$$

Remarks 4.1 i) Asumming that $p = \dim T > 2$ and $n = \dim M > 2$, the set (E_1) of the Einstein equations can be rewritten in the more natural form

$$(E_1') \qquad \begin{cases} H_{\alpha\beta} - \dfrac{H}{2}h_{\alpha\beta} = \mathcal{K}\tilde{\mathcal{T}}_{\alpha\beta} \\ R_{ij} - \dfrac{R}{2}g_{ij} = \mathcal{K}\tilde{\mathcal{T}}_{ij}, \end{cases}$$

where $\tilde{\mathcal{T}}_{AB}$, $A, B \in \{\alpha, i\}$ are the adapted local components of a new stress-energy d-tensor $\tilde{\mathcal{T}}$. This new form of the Einstein equations will be treated detalied in the more general case of a *generalized metrical multi-time Lagrange space* [9].

ii) It is remarkable that, writing the Einstein equations of metrical multi-time Lagrange space of electrodynamics in the new form (E_1'), these reduce to the classical ones.

Note that, in order to have the compatibility of the Einstein equations, it is necessary that the certain adapted local components of the stress-energy d-tensor vanish *"a priori"*.

At the same time, it is well known that, from a physical point of view, the stress-energy d-tensor $\mathcal{T}$ must verify the local *conservation laws* $\mathcal{T}^B_{A|B} = 0$, $\forall\ A \in \{\alpha, i, {}^{(\alpha)}_{(i)}\}$, where $\mathcal{T}^B_A = G^{BD}\mathcal{T}_{DA}$.

In these conditions, by computations, we obtain the following

Theorem 4.2 *The conservation laws of the Einstein equations of the gravitational h-potential of autonomous metrical multi-time Lagrange space reduce to*

$$\begin{cases} \left[H^\mu_\beta - \dfrac{H+r}{2}\delta^\mu_\beta\right]_{/\mu} = 0 \\ \left[r^m_j - \dfrac{H+r}{2}\delta^m_j\right]_{|m} = 0. \end{cases}$$

Remark 4.2 Taking into account the components $\tilde{\mathcal{T}}_{\alpha\beta}$ and $\tilde{\mathcal{T}}_{ij}$ of the new stress-energy d-tensor $\mathcal{T}$ appeared in the (E_1') form of the Einstein equations, the conservation laws modify in the following simple and natural form

$$\tilde{\mathcal{T}}^{\mu}_{\beta/\mu} = 0,\ \tilde{\mathcal{T}}^{m}_{j|m} = 0. \tag{4.1}$$

All these will be discussed detalied in [9].

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